Journal of Nonlinear Mathematical Physics 1998, V.5, N 4, 462–470. ARTICLE

Mode-Coupling and Nonlinear Landau Damping Effects in Auroral Farley-Buneman Turbulence

A.M. HAMZA

Department of Physics, Center for Space Research, University of New Brunswick, Fredericton, NB, E3A-5A3, Canada

Received June 2, 1998; Accepted September 01, 1998

Abstract

The fundamental problem of Farley-Buneman turbulence in the auroral E-region has been discussed and debated extensively in the past two decades. In the present paper we intend to clarify the different steps that the auroral E-region plasma has to undergo before reaching a steady state. The mode-coupling calculation, for Farley-Buneman turbulence, is developed in order to place it in perspective and to estimate its magnitude relative to the anomalous effects which arise through the nonlinear wave-particle interaction. This nonlinear effect, known as nonlinear "Landau damping" is due to the coupling of waves which produces other waves which in turn lose energy to the bulk of the particles by Landau damping. This leads to a decay of the wave energy and consequently a heating of the plasma. An equation governing the evolution of the field spectrum is derived and a physical interpration for each of its terms is provided.

1 Introduction

The selfconsistent theory of Farley-Buneman turbulence as developed by *Sudan* [8, 9], and by *Hamza and St-Maurice* [3, 4] has addressed successfully a number of features observed by radars in the equatorial and auroral ionosphere. Recently, *Hamza and St-Maurice* [5, 6] developed a general fluid formalism, taking into account the previously ignored nonlinear electron inertia effect term. The new turbulence theory provides a possible explanation for the existence of large aspect angle echoes in the auroral E-region, and justifies selfconsistently the link between electron heating along the magnetic field lines and the anomalous effects induced by perpendicular Farley-Buneman turbulence. In [5, 6] it is shown that the advection of parallel current by perpendicular drifts leads to an anomalous parallel collision frequency. In other words, it is shown that perpendicular Farley-Buneman turbulence can indeed lead to anomalous parallel effects which in turn allow for parallel electric fields to be generated. This has an impact on energy conservation since the zeroth order Ohmic heating which was confined to the plane perpendicular to the magnetic field, can now have a parallel component due to plasma wave turbulence

$\langle J_{e\|} E_{\|} \rangle$. This is made possible by the nonlinear electron inertia term. In order to clarify the physics related to the energy transport properties of the E-region plasma we have to discuss the different mechanisms of energy tranfer from the free energy source confined to the plane perpendicular to the magnetic field to the particles moving along the field lines. This can only be achieved by including mode-coupling effects and describing their role as it pertains to the final picture.

In what follows, we will use the results of *Hamza and St-Maurice* [5, 6] to establish the final link by identifying the mode-coupling terms and the terms associated with the analog of a wave-particle interaction respectively. In particular, we will stress the fact that the mode-coupling terms are energy conserving and that they do contribute a frequency broadening $\Delta\omega_{\mathbf{k}}$ as already shown by *Hamza and St-Maurice* [3] as well as *Sudan* [8, 9]. Finally we will discuss the roles of the mode-coupling terms and that of the anomalous effects generated via nonlinear wave-particle analogs respectively. However, more important than the comparison of the relative magnitudes of the mode-coupling rate and the parallel rate of dissipation generated by the anomalous effects is the difference in nature of the physics associated with each respectively. Although, the frequency broadening generated by mode-coupling, as shown by *Hamza and St-Maurice* [3], might indeed balance the linear growth rate of a Farley-Buneman wave, it is only shuffling energy into other waves. The parallel dissipation rate, on the other hand, allows wave energy to be channelled directly into electron kinetic energy. In steady state turbulence a physical process must ultimately convert wave energy into thermal energy. This energy conservation, in fact is a requirement for any selfconsistent theoretical model.

We already know that during the so-called “quasi-linear” phase, the background particle distribution diffuses in such a way as to bring the growth rate of the unstable waves to zero (“plateauing” of the distribution function), leaving a quasi-stationary spectrum interact in such a way as to distort the spectrum but keep the energy in the spectrum roughly constant. This is achieved by extracting energy from the resonant particles and dumping it unto the waves. Quasi-linear theory leads to the development of a quasi-equilibrium spectrum which persists indefinitely. The spectral energy can be shown to be proportional to γ/ω, where γ and ω represent the growth rate and the eigenfrequency respectively. This energy is small since it is proportional to $\gamma/\omega \ll 1$, and therefore can not be considered to be an ultimate saturation mechanism for the problem at hand as shown by *St-Maurice* [7]. Mode-coupling effects, neglected by the quasi-linear theory lead to a broadening of the equlibrium spectrum. The mode-coupling effects can be divided into two categories. The first class of terms can be qualified as “resonant” in the sense that waves couple while satisfying the following relations (frequency and wavemomentum matching)

$$\begin{aligned} &\omega_{\mathbf{k}'} + \omega_{\mathbf{k}''} = \omega_{\mathbf{k}}, \\ &\mathbf{k}' + \mathbf{k}'' = \mathbf{k}. \end{aligned} \tag{1}$$

The resonant mode coupling terms lead to a broadening of the quasi-linear spectrum and the time scale associated with this effect is proportional to the spectral density (in our case $|\delta\phi|^2 \propto \gamma$), and therefore the resonant mode coupling process is on the same time scale as the quasi-linear time scale. However, as shown by *Hamza and St-Maurice* [3, 4] and *Drummond and Pines* [2] before them, the resonant mode coupling leaves the total energy in the wave spectrum virtually untouched (constant).

One major constraint in resonant mode coupling is expressed by equation (1). The driving term is a quadratic nonlinearity of the form $\phi_{\mathbf{k}'}(t)\Phi_{\mathbf{k}-\mathbf{k}'}(t)$ and induces a time dependence of $\exp\left[i(\omega_{\mathbf{k}'}+\omega_{\mathbf{k}-\mathbf{k}'})t\right]$, which in general can be very different from the time dependence of $\phi_{\mathbf{k}}(t)$, that is, $\exp\left[i\omega_{\mathbf{k}}t\right]$. In other words, in the general case one has

$$\begin{aligned} &\omega_{\mathbf{k}'}+\omega_{\mathbf{k}''}\neq\omega_{\mathbf{k}}, \\ &\mathbf{k}^{'}+\mathbf{k}^{''}=\mathbf{k}. \end{aligned} \tag{2}$$

The nonlinear source forces the field $\phi_{\mathbf{k}}(t)$ to have terms with the time dependence of the source rather than the natural time dependence of the field itself. This is very much the case for a forced harmonic oscillator. This leads to separating the field variable into two parts. The first part gives rise to resonant coupling because of frequency matching, while the second part arises because of the non-resonant parts of the nonlinear source. An essential feature of the latter part is that $(\omega_{\mathbf{k}'}+\omega_{\mathbf{k}''})$ can be fairly small so as to allow the phase velocity of the resultant wave $(\omega_{\mathbf{k}'}+\omega_{\mathbf{k}-\mathbf{k}'})/k_\parallel$ to be smaller than the electron thermal velocity, and for that matter the ion thermal velocity. This means that the non-resonant contribution of the mode-coupling terms can undergo Landau damping. The non-resonant waves Landau damp and in the process give their energy to both the electron and ions, i.e. heat the plasma along the field lines.

The main objective of the present calculation, is to show that the nonlinearities in Farley-Buneman turbulence contribute coherent and resonant terms. These terms are isolated and shown to have different impacts on the physics, especially on the energy conservation constraint. We have omitted to discuss transient phenomena because we are only interested in the global energy conservation on a long time scale. Higher order nonlinearities have also been neglected in the perturbation analysis. The model has been extensively described in [6]. Therefore we will refer the reader to that paper for formal manipulations and for details related to the derivation of certain fundamental results.

2 Theoretical Model

The fundamental assumptions and the details of the theoretical model are described, as mentioned above in [6]. We will only quote the results neccessary for the development of an energy balance equation containing all the relevant terms including the mode-coupling terms.

The starting point of this paper is equation (24) of [6] which can be written in a compact form as follows

$$-in_0\frac{e\phi_{\mathbf{k}\omega}}{T_e}\left(\omega-\omega_{\mathbf{k}}^{(L)}\right)=(1)+(2)+(3), \tag{3}$$

where the three terms on the right hand side of equation (3) can be explicitly written as

$$(1)\equiv-\frac{i}{1+\psi_{\mathbf{k}\omega}}\frac{m_i}{e^2}\frac{\omega^2-k_\perp^2\frac{T_i}{m_i}+i\nu_i\omega}{k_\perp^2}\frac{k_\parallel}{\Lambda_{\mathbf{k}\omega}}\sum_{\mathbf{k}'\omega'}A_{\mathbf{k};\mathbf{k}'}^{\omega;\omega'}\frac{e\phi_{\mathbf{k}'\omega'}}{T_e}J^{\parallel}_{e\mathbf{k}-\mathbf{k}'\omega-\omega'}. \tag{4}$$

This term represents the advection of parallel current by perpendicular and parallel drifts through electron inertia. The second term corresponds to the advection of perpendicular

current by parallel and perpendicular drifts and can be written in the following form

$$(2) \equiv \frac{i}{1+\psi_{\mathbf{k}\omega}} \frac{m_i}{e^2} \frac{\omega^2 - k_\perp^2 \frac{T_i}{m_i} + i\nu_i\omega}{k_\perp^2} \frac{\Omega_e}{\Omega_e^2 + \Lambda_{\mathbf{k}\omega}^2}$$
$$\times \sum_{\mathbf{k}'\omega'} \left(\frac{\Omega_e}{\Lambda_{\mathbf{k}-\mathbf{k}'\omega-\omega'}} - \frac{\Lambda_{\mathbf{k}\omega}}{\Omega_e} \right) A_{\mathbf{k};\mathbf{k}'}^{\omega;\omega'} \frac{e\phi_{\mathbf{k}'\omega'}}{T_e} \mathbf{k}' erp \cdot \mathbf{J}_{e\mathbf{k}-\mathbf{k}'\omega-\omega'}^{\perp}. \tag{5}$$

It is important to emphasize the fact that these two terms were completely omitted by previous theories addressing the issues related to Farley-Buneman turbulence in the auroral as well as equatorial E region.

Finally the last term represents the classical mode coupling terms, which can be written in the following form

$$(3) \equiv -\frac{n_0}{1+\psi_{\mathbf{k}\omega}} \frac{\Omega_e}{\Omega_e^2+\Lambda_{\mathbf{k}\omega}^2} \sum_{\mathbf{k}'\omega'} b_{\mathbf{k};\mathbf{k}'}^{\omega;\omega'} \left\{ \hat{\mathbf{z}}\cdot(\mathbf{k}\times\mathbf{k}') - \frac{\Lambda_{\mathbf{k}\omega}}{\Omega_e} \mathbf{k}'_\perp \cdot \mathbf{k}_\perp \left(1 + \frac{k_\parallel^2}{k_\perp^2} \frac{\Omega_e^2+\Lambda_{\mathbf{k}\omega}^2}{\Lambda_{\mathbf{k}\omega}^2} \right) \right\}$$
$$\times V_{the}^2 \frac{e\phi_{\mathbf{k}'\omega'}}{T_e} \frac{e\phi_{\mathbf{k}-\mathbf{k}'\omega-\omega'}}{T_e} - n_0 \frac{\Omega_e}{1+\psi_{\mathbf{k}\omega}} \frac{\Omega_e^2}{\Omega_e^2+\Lambda_{\mathbf{k}\omega}^2} \tag{6}$$
$$\times \sum_{\mathbf{k}'\omega'} \frac{A_{\mathbf{k};\mathbf{k}'}^{\omega;\omega'}}{\Lambda_{\mathbf{k}-\mathbf{k}'\omega-\omega'}} b_{\mathbf{k};\mathbf{k}'}^{\omega;\omega'} c_{\mathbf{k};\mathbf{k}'}^{\omega;\omega'} \frac{\hat{\mathbf{z}}\cdot(\mathbf{k}\times\mathbf{k}')}{|\mathbf{k}_\perp - \mathbf{k}'_\perp|} \frac{e\phi_{\mathbf{k}'\omega'}}{T_e} \phi_{\mathbf{k}-\mathbf{k}'\omega-\omega'}.$$

In the above expressions we have

$$\psi_{\mathbf{k}\omega} = \frac{\Omega_e}{\Omega_i} \frac{\nu_i \Lambda_{\mathbf{k}\omega}}{\Omega_e^2+\Lambda_{\mathbf{k}\omega}^2} \left(1 + \frac{k_\parallel^2}{k_\perp^2} \frac{\Omega_e^2+\Lambda_{\mathbf{k}\omega}^2}{\Lambda_{\mathbf{k}\omega}^2} \right) \tag{7}$$

while $\omega_{\mathbf{k}}^{(L)}$ represents the linear eigenfrequency for Farley-Buneman waves. The coefficients $A_{\mathbf{k};\mathbf{k}'}^{\omega;\omega'}$, $b_{\mathbf{k};\mathbf{k}'}^{\omega;\omega'}$, and $c_{\mathbf{k};\mathbf{k}'}^{\omega;\omega'}$ are defined in the paper [6], and the parallel and perpendicular currents $J_e^{\parallel}$ and $\mathbf{J}_e^{\perp}$ are given by equations (21) and (22) in [6] respectively.

It is clear from equation (6) that when the nonlinear right hand side is neglected we recover the classic Farley-Buneman dispersion relation.

The first two terms on the right hand side of equation (3) contribute the coherent terms discussed extensively in the paper [6]. The coherent terms can easily be identified through a formal substitution of the expressions for $J_e^{\parallel}$ and $\mathbf{J}_e^{\perp}$ into equation (3). Isolating these coherent terms, while neglecting the four-wave coupling which generates cubic nonlinearities, allows us to rewrite the top equation (6) in the following form. To illustrate this iteration process let us consider the first term in equation (3). To proceed we need the expression for the parallel current

$$J_{e\mathbf{k}\omega}^{\parallel} = -i\frac{n_0 e^2}{m_e} \frac{k_\parallel}{\Lambda_{\mathbf{k}\omega}} F(\omega, k_\perp)\phi_{\mathbf{k}\omega} - \sum_{\mathbf{k}'\omega'} \frac{A_{\mathbf{k};\mathbf{k}'}^{\omega;\omega'}}{\Lambda_{\mathbf{k}\omega}} \phi_{\mathbf{k}'\omega'} J_{e\mathbf{k}-\mathbf{k}'\omega-\omega'}^{\parallel}$$
$$-i\frac{e^2}{m_e} \sum_{\mathbf{k}'\omega'} \frac{k'_\parallel}{\Lambda_{\mathbf{k}\omega}} \phi_{\mathbf{k}'\omega'} \delta n_{\mathbf{k}-\mathbf{k}'\omega-\omega'}. \tag{8}$$

Substituting equation (8) into equation (4) one obtains two types of terms. The first type can be combined with the mode coupling term given by equation (5), while the second type of term, a cubic term, is of the form $\phi\phi J_e^{\|}$. A second iteration of this term leads to a cubic term of the form $|\phi_{\mathbf{k}'\omega'}|^2\phi_{\mathbf{k}\omega}$. This term can combined with the left hand side of equation (3) to obtain a renormalized dispersion relation. This effect introduces an anomalous frequency solely due to nonlinear effects analog to a higher order wave-particle interaction.

$$- i(\omega - \omega_{\mathbf{k}})\phi_{\mathbf{k}\omega} = \sum_{\mathbf{k}'\omega'} M_{\mathbf{k};\mathbf{k}'}^{\omega;\omega'} \phi_{\mathbf{k}'\omega'}\phi_{\mathbf{k}-\mathbf{k}'\omega-\omega'}, \tag{9}$$

where $M_{\mathbf{k};\mathbf{k}'}^{\omega;\omega'}$ is the nonlinear coupling coefficient obtained by substituting the expressions for $J_e^{\|}$ and $\mathbf{J}_e^{\perp}$, as given in [6], in equation (3) above. Where the new frequency $\omega_{\mathbf{k}}$ includes the coherent terms generated by the nonlinearities, and written as $\nu^{\|}J_e^{\|}$ and $\nu^{\perp}\mathbf{J}_e^{\perp}$, where $J_e^{\|}$ and $\mathbf{J}_e^{\perp}$ are shown to be propotional to $\phi_{\mathbf{k}\omega}$ to first order. These nonlinear coherent expressions were derived and discussed extensively in [6]. $\omega_{\mathbf{k}}$ can now be written as follows:

$$\omega_{\mathbf{k}} = \omega_{\mathbf{k}}^{(L)} + \omega_{\mathbf{k}}^{C}, \tag{10}$$

where $\omega_{\mathbf{k}}^{C}$ represents the anomalous (coherent) contribution by the nonlinear wave-particle analogs in equation (6). As one can expect, the nonlinear expression for this anomalous frequency is quite complicated and will not be given here.

Making the following change in wavevectors and frequencies $\mathbf{k}' \rightarrow \mathbf{k} - \mathbf{k}'$, and $\omega' \rightarrow \omega - \omega'$ leads to rewriting equation (9) in a useful form for our purpose as will become clear in what follows.

$$- i(\omega - \omega_{\mathbf{k}})\phi_{\mathbf{k}\omega} = \frac{1}{2}\sum_{\mathbf{k}'\omega'} \left(M_{\mathbf{k};\mathbf{k}'}^{\omega;\omega'} + M_{\mathbf{k};\mathbf{k}-\mathbf{k}'}^{\omega;\omega-\omega'} \right) \phi_{\mathbf{k}'\omega'}\phi_{\mathbf{k}-\mathbf{k}'\omega-\omega'}. \tag{11}$$

Multiplying by $\phi^*_{\mathbf{k}\omega}$ and ensemble averaging leads to the evolution equation for the spectral density $\langle|\phi_{\mathbf{k}\omega}|^2\rangle$

$$- i(\omega - \omega_{\mathbf{k}})\langle|\phi_{\mathbf{k}\omega}|^2\rangle = \frac{1}{2}\sum_{\mathbf{k}'\omega'} \left(M_{\mathbf{k};\mathbf{k}'}^{\omega;\omega'} + M_{\mathbf{k};\mathbf{k}-\mathbf{k}'}^{\omega;\omega-\omega'} \right) \left\langle \phi_{\mathbf{k}'\omega'}\phi_{\mathbf{k}-\mathbf{k}'\omega-\omega'}\phi^*_{\mathbf{k}\omega} \right\rangle. \tag{12}$$

At this stage one has to close the system by prescribing the triple correlation function. We shall assume that the distribution of potential fluctuations is Gaussian, which allows to evaluate explicitly, via a perturbation scheme, the right hand side of equation (12), that is

$$\phi = \phi^{(1)} + \phi^{(2)} + \cdots$$

with

$$\begin{aligned} &-i\left(\omega - \omega_{\mathbf{k}}^{(L)}\right)\phi_{\mathbf{k}\omega}^{(1)} = 0, \\ &-i(\omega - \omega_{\mathbf{k}})\phi_{\mathbf{k}\omega}^{(2)} = \frac{1}{2}\sum_{\mathbf{k}'\omega'} \left(M_{\mathbf{k};\mathbf{k}'}^{\omega;\omega'} + M_{\mathbf{k};\mathbf{k}-\mathbf{k}'}^{\omega;\omega-\omega'} \right) \phi_{\mathbf{k}'\omega'}^{(1)}\phi_{\mathbf{k}-\mathbf{k}'\omega-\omega'}^{(1)} \end{aligned} \tag{13}$$

the triple correlation function can be written as:

$$\begin{aligned}\left\langle \phi_{\mathbf{k}'\omega'}\phi_{\mathbf{k}-\mathbf{k}'\omega-\omega'}\phi^*_{\mathbf{k}\omega}\right\rangle &= \left\langle \phi^{(1)}_{\mathbf{k}'\omega'}\phi^{(1)}_{\mathbf{k}-\mathbf{k}'\omega-\omega'}\phi^{(1)*}_{\mathbf{k}\omega}\right\rangle + \left\langle \phi^{(1)}_{\mathbf{k}'\omega'}\phi^{(1)}_{\mathbf{k}-\mathbf{k}'\omega-\omega'}\phi^{(2)*}_{\mathbf{k}\omega}\right\rangle \\ &+ \left\langle \phi^{(1)}_{\mathbf{k}'\omega'}\phi^{(2)}_{\mathbf{k}-\mathbf{k}'\omega-\omega'}\phi^{(1)*}_{\mathbf{k}\omega}\right\rangle + \left\langle \phi^{(2)}_{\mathbf{k}'\omega'}\phi^{(1)}_{\mathbf{k}-\mathbf{k}'\omega-\omega'}\phi^{(1)*}_{\mathbf{k}\omega}\right\rangle .\end{aligned} \tag{14}$$

The first term on the right hand side of equation (14) vanishes, while the others can be combined with equation (13) to lead to

$$\begin{aligned}\left\langle \phi^{(1)}_{\mathbf{k}'\omega'}\phi^{(1)}_{\mathbf{k}-\mathbf{k}'\omega-\omega'}\phi^{(2)*}_{\mathbf{k}\omega}\right\rangle &= -\frac{i}{2}\sum_{\mathbf{k}'\omega'}\frac{\left(M^{\omega;\omega'}_{\mathbf{k};\mathbf{k}'}+M^{\omega;\omega-\omega'}_{\mathbf{k};\mathbf{k}-\mathbf{k}'}\right)^*}{(\omega-\omega_{\mathbf{k}})}\langle|\phi_{\mathbf{k}'\omega'}|^2\rangle\langle|\phi_{\mathbf{k}-\mathbf{k}'\omega-\omega'}|^2\rangle, \\ \left\langle \phi^{(1)}_{\mathbf{k}'\omega'}\phi^{(2)}_{\mathbf{k}-\mathbf{k}'\omega-\omega'}\phi^{(1)*}_{\mathbf{k}\omega}\right\rangle &= \frac{i}{2}\sum_{\mathbf{k}'\omega'}\frac{\left(M^{\omega-\omega';-\omega'}_{\mathbf{k}-\mathbf{k}';-\mathbf{k}'}+M^{\omega-\omega';\omega}_{\mathbf{k}-\mathbf{k}';\mathbf{k}}\right)}{(\omega-\omega'-\omega_{\mathbf{k}-\mathbf{k}'})}\langle|\phi_{\mathbf{k}'\omega'}|^2\rangle\langle|\phi_{\mathbf{k}\omega}|^2\rangle, \\ \left\langle \phi^{(2)}_{\mathbf{k}'\omega'}\phi^{(1)}_{\mathbf{k}-\mathbf{k}'\omega-\omega'}\phi^{(1)*}_{\mathbf{k}\omega}\right\rangle &= \frac{i}{2}\sum_{\mathbf{k}'\omega'}\frac{\left(M^{\omega';\omega'-\omega}_{\mathbf{k}';\mathbf{k}'-\mathbf{k}}+M^{\omega';\omega}_{\mathbf{k}';\mathbf{k}}\right)}{(\omega'-\omega_{\mathbf{k}'})}\langle|\phi_{\mathbf{k}'\omega'}|^2\rangle\langle|\phi_{\mathbf{k}\omega}|^2\rangle.\end{aligned} \tag{15}$$

This finally allows us to write the equation (12) for the spectral density in the following form:

$$\begin{aligned}-i(\omega-\omega_{\mathbf{k}})\langle|\phi_{\mathbf{k}\omega}|^2\rangle &= \frac{1}{2}\sum_{\mathbf{k}'\omega'}\left(M^{\omega;\omega'}_{\mathbf{k};\mathbf{k}'}+M^{\omega;\omega-\omega'}_{\mathbf{k};\mathbf{k}-\mathbf{k}'}\right) \\ &\times\left\{\frac{i}{(\omega-\omega'-\omega_{\mathbf{k}-\mathbf{k}'})}\left(M^{\omega-\omega';-\omega'}_{\mathbf{k}-\mathbf{k}';-\mathbf{k}'}+M^{\omega-\omega';\omega}_{\mathbf{k}-\mathbf{k}';\mathbf{k}}\right)\langle|\phi_{\mathbf{k}'\omega'}|^2\rangle\langle|\phi_{\mathbf{k}\omega}|^2\rangle\right. \\ &\left.-\frac{1}{2}\frac{i}{(\omega-\omega_{\mathbf{k}})}\left(M^{\omega;\omega'}_{\mathbf{k};\mathbf{k}'}+M^{\omega;\omega-\omega'}_{\mathbf{k};\mathbf{k}-\mathbf{k}'}\right)^*\langle|\phi_{\mathbf{k}'\omega'}|^2\rangle\langle|\phi_{\mathbf{k}-\mathbf{k}'\omega-\omega'}|^2\rangle\right\}.\end{aligned} \tag{16}$$

The first term on the right hand side corresponds to a broadening due to resonant three-wave coupling (mode coupling), and can be incorporated with the left hand side to give

$$\begin{aligned}&-i(\omega-\omega_{\mathbf{k}}-\Delta\omega_{\mathbf{k}})\langle|\phi_{\mathbf{k}\omega}|^2\rangle \\ &\quad= -\frac{1}{4}\frac{i}{(\omega-\omega_{\mathbf{k}})}\sum_{\mathbf{k}'\omega'}\left(M^{\omega;\omega'}_{\mathbf{k};\mathbf{k}'}+M^{\omega;\omega-\omega'}_{\mathbf{k};\mathbf{k}-\mathbf{k}'}\right)^*\langle|\phi_{\mathbf{k}'\omega'}|^2\rangle\langle|\phi_{\mathbf{k}-\mathbf{k}'\omega-\omega'}|^2\rangle,\end{aligned} \tag{17}$$

where

$$\Delta\omega_{\mathbf{k}} = -\frac{1}{2}\sum_{\mathbf{k}'\omega'}\frac{\left(M^{\omega;\omega'}_{\mathbf{k};\mathbf{k}'}+M^{\omega;\omega-\omega'}_{\mathbf{k};\mathbf{k}-\mathbf{k}'}\right)\left(M^{\omega-\omega';-\omega'}_{\mathbf{k}-\mathbf{k}';-\mathbf{k}'}+M^{\omega-\omega';\omega}_{\mathbf{k}-\mathbf{k}';\mathbf{k}}\right)}{(\omega-\omega'-\omega_{\mathbf{k}-\mathbf{k}'})}\langle|\phi_{\mathbf{k}'\omega'}|^2\rangle. \tag{18}$$

These results are very much similar to the two dimensional results published by *Hamza and St-Maurice* [4] when looking explicitly at the two dimensional Farley-Buneman turbulence. The coefficients in the current case are more complicated because we have taken into account the electron inertia nonlinearities and included the parallel effects. We do not

intend to explicitly write the expressions for the coefficients since, as might be expected they are quite complicated and would not serve the purpose of this paper. However, we have in equation (17) isolated two types of terms. The first term is a coherent analog of a nonlinear wave-particle interaction, nonlinear Landau-damping, while the second is due to a resonant three-wave coupling. The term on the right hand side of equation (17) is due solely to mode coupling.

At this stage, one can study two limiting cases, namely a strong turbulence limit and a weak tubulence one.

2.1 Strong Turbulence Limit

In this case equation (17) can be written in a more familiar form:

$$\langle|\phi_{\mathbf{k}\omega}|^2\rangle = \frac{1}{4}\frac{1}{(\omega-\omega_{\mathbf{k}}-\Delta\omega_{\mathbf{k}})^2}\sum_{\mathbf{k}'\omega'}\left(M^{\omega;\omega'}_{\mathbf{k};\mathbf{k}'}+M^{\omega;\omega-\omega'}_{\mathbf{k};\mathbf{k}-\mathbf{k}'}\right)^*\langle|\phi_{\mathbf{k}'\omega'}|^2\rangle\langle|\phi_{\mathbf{k}-\mathbf{k}'\omega-\omega'}|^2\rangle. \quad (19)$$

This equation can be written in the following form

$$\langle|\phi_{\mathbf{k}\omega}|^2\rangle = \frac{\langle|\tilde{\phi}_{\mathbf{k}\omega}|^2\rangle}{|\epsilon_{\mathbf{k}\omega}|^2}, \quad (20)$$

where $\langle|\tilde{\phi}_{\mathbf{k}\omega}|^2\rangle$ is the incoherent driven potential obtained by identifying the right hand side of equation (20) to the right hand side of equation (19). This has been the object of a long debated issue related to describing strong turbulence as it occurs in plasmas. We will try to illustrate, very briefly, the results of the debate on strong turbulence using the fundamental results of *Boutros-Ghali and Dupree* [1] and references therein. When writing Poisson's equation we separate the coherent (phase coherent with the electric field) from the incoherent contribution of the distribution function in the following way

$$\nabla\cdot\delta\mathbf{E} = 4\pi nq\int d\mathbf{v}\left(\delta f^c+\delta\tilde{f}\right), \quad (21)$$

where δf^c and $\delta\tilde{f}$ represent the coherent and incoherent parts of the particle distribution function respectively. The coherent part contributes to the dispersion relation which allows us to write

$$i\epsilon_{\mathbf{k}\omega}\mathbf{k}\cdot\delta\mathbf{E} = 4\pi nq\int d\mathbf{v}\delta\tilde{f} \quad (22)$$

using $\delta\mathbf{E}=-i\mathbf{k}\phi_{\mathbf{k}\omega}$ leads to

$$\phi_{\mathbf{k}\omega} = \frac{4\pi nq}{k^2\epsilon_{\mathbf{k}\omega}}\int d\mathbf{v}\delta\tilde{f} = \frac{\tilde{\phi}_{\mathbf{k}\omega}}{\epsilon_{\mathbf{k}\omega}} \quad (23)$$

which basically leads to equation (20). A strong turbulence result (see for example *Boutros-Ghali and Dupree* [1]).

2.2 Weak Turbulence Case

In the weak turbulence regime, in which the waves are weakly growing, with small line widths $Im\,(\omega_{\mathbf{k}})$, and replacing Fourier series with Fourier integrals, we obtain

$$\sum_{\mathbf{k}\omega}\langle|\phi_{\mathbf{k}\omega}|^2\rangle \rightarrow \int \frac{d\mathbf{k}}{(2\pi)^4}\int d\omega \langle\phi^2\rangle_{\mathbf{k}\omega} \tag{24}$$

and using the following spectral relation

$$\langle\phi^2\rangle_{\mathbf{k}\omega} = \langle\phi^2\rangle_{\mathbf{k}} 2\pi\delta(\omega-\omega_{\mathbf{k}}). \tag{25}$$

Let us now go back to equation (17) and integrate over the frequency using the following identity

$$-\,2Re\int\frac{d\omega}{2\pi}i(\omega-\omega_{\mathbf{k}}^{(L)})\langle|\phi_{\mathbf{k}\omega}|^2\rangle = \frac{d}{dt}\langle\phi^2\rangle_{\mathbf{k}}. \tag{26}$$

In other words, the time derivative represents the quasi-linear growth rate. We finally obtain

$$\begin{aligned}\frac{d}{dt}\langle\phi^2\rangle_{\mathbf{k}} &= -2Re\int\frac{d\omega}{2\pi}i(\omega_{\mathbf{k}}^{C}+\Delta\omega_{\mathbf{k}})\langle|\phi_{\mathbf{k}\omega}|^2\rangle\\ &\quad -2Re\int\frac{d\mathbf{k}'}{(2\pi)^3}\left(M_{\mathbf{k};\mathbf{k}'}+M_{\mathbf{k};\mathbf{k}-\mathbf{k}'}\right)^*\langle\phi^2\rangle_{\mathbf{k}'}\langle\phi^2\rangle_{\mathbf{k}-\mathbf{k}'}\pi\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-\omega_{\mathbf{k}-\mathbf{k}'})\end{aligned} \tag{27}$$

the first two terms contribute two different growth rates: $\gamma_{\mathbf{k}}^{C}$ due to the coherent nonlinear wave-particle analog, and $\gamma_{\mathbf{k}}^{\Delta}$ due to mode coupling effects. This allows us to write the following spectral equation

$$\begin{aligned}\frac{d}{dt}\langle\phi^2\rangle_{\mathbf{k}} &= -2(\gamma_{\mathbf{k}}^{C}+\gamma_{\mathbf{k}}^{\Delta})\langle\phi^2\rangle_{\mathbf{k}}\\ &\quad -2Re\int\frac{d\mathbf{k}'}{(2\pi)^3}\left(M_{\mathbf{k};\mathbf{k}'}+M_{\mathbf{k};\mathbf{k}-\mathbf{k}'}\right)^*\langle\phi^2\rangle_{\mathbf{k}'}\langle\phi^2\rangle_{\mathbf{k}-\mathbf{k}'}\pi\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'}-\omega_{\mathbf{k}-\mathbf{k}'})\end{aligned} \tag{28}$$

integrating over $\mathbf{k}$ one can show that the mode-coupling term conserves the total mode energy, which leaves us with:

$$\int d\mathbf{k}\frac{d}{dt}\langle\phi^2\rangle_{\mathbf{k}} = -2\int d\mathbf{k}(\gamma_{\mathbf{k}}^{C})\langle\phi^2\rangle_{\mathbf{k}} \tag{29}$$

in other words, in order to reach a steady state the right hand side has to vanish. This leads us to the following interpretation. A steady state requires both resonant mode coupling and the analog of a wave-particle exchange to take place; one in the absence of the other does not neccessarily guarantee steady state turbulence. Consequently, a steady state in Farley-Buneman turbulence requires that the free energy available in the plane perpendicular to the magnetic field, extracted via a linear instability, be converted into particle kinetic energy along the field lines via an analog of a nonlinear wave-particle mechanism.

3 Summary and Conclusions

In this letter we have established the neccessary condition for steady state Farley-Buneman turbulence. We have shown that in order to reach steady state the growth in the plane perpendicular to the magnetic field has to be compensated by a growth along the field line; A perfect analogy with a leaky bucket, if the we want the water level in the bucket to remain unchanged then we need to balance the rate of pouring to the rate of leaking water. In other words free energy is extracted from the drifts by the Farley-Buneman waves which start to grow and couple. This process will go on, and no steady state will be reached unless the energy is given back to the particles, in this case the particles moving along the field lines. So two processes take place. First, linear growth and resonant coupling of Farley-Buneman waves, then a nonlinear wave particle coupling along the field lines. These two mechanisms when combined together can lead to steady state turbulence.

Acknowledgments

The author is grateful to J.-P. St-Maurice and D.R. Moorcroft for very useful discussions, and thanks D. Montgomery for the private communication. Funding for this research has been provided by the Canadian Space Agency.

References

[1] Boutros-Ghali T. and Dupree T.H., Theory of two-point correlation function in a Vlasov plasma, *Phys. Fluids*, 1981, V.24.

[2] Drummond W.E. and Pines D., Nonlinear stability of plasma oscillations, *Nuclear Fusion*, 1962.

[3] Hamza A.M. St-Maurice J.-P., A turbulent theoretical framework for the study of current-driven E region irregularities at high latitudes: Basic derivation and application to gradient-free situations, *J. Geophys. Res.*, 1993, V.98.

[4] Hamza A.M. and St-Maurice J.-P., A self-consistent fully turbulent theory of auroral E region irregularities, *J. Geophys. Res.*, 1993, V.98.

[5] Hamza A.M. and St-Maurice J.-P., Large aspect angles in auroral E region echoes: A self-consistent turbulent fluid theory, *J. Geophys. Res.*, 1995.

[6] Hamza A.M. and St-Maurice J.P., A fully self-consistent fluid theory of anomalous transport in Farley-Buneman turbulence, *J. Geophys. Res.*, 1995.

[7] St-Maurice J.-P., Electron heating by plasma waves in the high latitude E-region and related effects: Theory, *Adv. Space Res.*, 1990, V.6.

[8] Sudan R.N., Nonlinear theory of type I irregularities in the equatorial electrojet, *Geophys. Res. Lett.*, 1983, V.10.

[9] Sudan R.N., Unified theory of type I and type II irregularities in the equatorial electrojet, *J. Geophys. Res.*, 1983, V.88.